Limit 2

1. Evaluate $\lim_{x\to 0}\frac{1-\cos 3x}{x}$.

Method 1

$$\lim_{x\to 0}\frac{1-\cos 3x}{x}=\lim_{x\to 0}\frac{(1+\cos 3x)(1-\cos 3x)}{x(1+\cos 3x)}=\lim_{x\to 0}\frac{1-\cos^2 3x}{x(1+\cos 3x)}=\lim_{x\to 0}\frac{\sin^2 3x}{x(1+\cos 3x)}$$

$$=3\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)(\sin 3x)\left(\frac{1}{1+\cos 3x}\right)=3\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\lim_{x\to 0}(\sin 3x)\lim_{x\to 0}\left(\frac{1}{1+\cos 3x}\right)=3(1)(0)\left(\frac{1}{1+1}\right)$$

$$=\underline{\underline{0}}$$

Method 2

$$\lim_{x\to 0}\frac{1-\cos 3x}{x}=\lim_{x\to 0}\frac{2\sin^2\frac{3x}{2}}{x(1+\cos 3x)}=\frac{9}{2}\lim_{x\to 0}x\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)^2\frac{1}{1+\cos 3x}$$

$$=\frac{9}{2}\lim_{x\to 0}x\ \lim_{x\to 0}\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)^2\lim_{x\to 0}\frac{1}{1+\cos 3x}=\frac{9}{2}(0)(1)^2\left(\frac{1}{1+1}\right)=\underline{\underline{0}}$$

Method 3 (L'Hôpital's rule)

$$\lim_{x\to 0}\frac{1-\cos 3x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos 3x)}{\frac{d}{dx}x}=\lim_{x\to 0}\frac{3\sin 3x}{1}=\underline{\underline{0}}$$

2. Evaluate $\lim_{x\to\infty}\frac{6x^4+\sin x}{x^4+1}$.

Since $-1\le\sin x\le 1$

$$-\frac{1}{x^4}\le\frac{\sin x}{x^4}\le\frac{1}{x^4}$$

$$-\lim_{x\to\infty}\frac{1}{x^4}\le\lim_{x\to\infty}\frac{\sin x}{x^4}\le\lim_{x\to\infty}\frac{1}{x^4}$$

By the Squeeze Principle, $\lim_{x\to\infty}\frac{\sin x}{x^4}=0$

$$\lim_{x\to\infty}\frac{6x^4+\sin x}{x^4+1}=\lim_{x\to\infty}\frac{6+\frac{\sin x}{x^4}}{1+\frac{1}{x^4}}=\frac{6+0}{1+0}=\underline{\underline{6}}$$

3. Evaluate $\lim_{x\to 0}\sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right]$.

$0 \le \sin^2\left(\frac{2\pi}{x}\right) \le 1$

$1 \le 1+\sin^2\left(\frac{2\pi}{x}\right) \le 2$

$\sqrt{x} \le \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] \le 2\sqrt{x}$

$\lim_{x\to 0}\sqrt{x} \le \lim_{x\to 0}\sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] \le 2\lim_{x\to 0}\sqrt{x}$

By the Squeeze Principle, $\lim_{x\to 0}\sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] = \underline{\underline{0}}$

4. Find $\lim_{n\to+\infty}\frac{1^1+2^2+3^3+\cdots+(n-1)^{n-1}+n^n}{n^n}$.

Let $L = \frac{1^1+2^2+3^3+\cdots+(n-1)^{n-1}+n^n}{n^n}$

$L > \frac{0+0+0+\cdots+0+n^n}{n^n} = 1 \Longrightarrow \lim_{n\to\infty} L \ge 1$

$L < \frac{n^1+n^2+n^3+\cdots+n^{n-1}+n^n}{n^n} = \frac{n\frac{n^n-1}{n-1}}{n^n} = \frac{n^n-1}{n^{n-1}(n-1)} = \frac{n^n-1}{n^n-n^{n-1}} = \frac{1-\frac{1}{n^n}}{1-\frac{1}{n}} \Longrightarrow \lim_{n\to\infty} L \le \lim_{n\to\infty}\frac{1-\frac{1}{n^n}}{1-\frac{1}{n}} = 1$

By the Squeeze Principle, $\lim_{n\to+\infty} L = \underline{\underline{1}}$

5. Evaluate $\lim_{x\to 0^+}(\sin x)^{\frac{1}{\ln x}}$.

Let $L = (\sin x)^{\frac{1}{\ln x}}$

$\ln L = \frac{\ln\sin x}{\ln x}$

$\lim_{x\to 0^+}\ln L = \lim_{x\to 0^+}\frac{\ln\sin x}{\ln x} = \lim_{x\to 0^+}\frac{\frac{d}{dx}(\ln\sin x)}{\frac{d}{dx}(\ln x)}$ ($\frac{\infty}{\infty}$ form, L'hopital Rule)

$= \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{x\cos x}{\sin x} = \frac{\lim_{x\to 0^+}\cos x}{\lim_{x\to 0^+}\frac{\sin x}{x}} = 1$

$\lim_{x\to 0^+} L = e^1 = \underline{\underline{e}}$

6. Evaluate $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$.

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right)=\int_0^1\frac{1}{1+x}dx$$

$$=\int_{x=0}^{x=1}\frac{1}{1+x}d(1+x)=\ln(1+x)\Big]_0^1=\underline{\underline{\ln 2}}$$

7. Evaluate $\lim_{n\to-\infty}\frac{3n}{\sqrt{n^2+1}}$.

Let $m=-n$

$$\lim_{n\to-\infty}\frac{3n}{\sqrt{n^2+1}}=\lim_{m\to+\infty}\frac{3(-m)}{\sqrt{(-m)^2+1}}=-\lim_{m\to+\infty}\frac{3m}{\sqrt{m^2+1}}=-\lim_{m\to+\infty}\frac{3}{\sqrt{1+\frac{1}{m^2}}}=\underline{\underline{-3}}$$

8. Find $\lim_{n\to+\infty}\left(\frac{2}{7^2}+\frac{2^3}{7^3}+\frac{2^5}{7^4}+\cdots+\frac{2^{2n-1}}{7^{n+1}}\right)$.

$$\lim_{n\to+\infty}\left(\frac{2}{7^2}+\frac{2^3}{7^3}+\frac{2^5}{7}+\cdots+\frac{2^{2n-1}}{7^{n+1}}\right)=\lim_{n\to+\infty}\frac{\frac{1}{2}}{7}\left(\frac{2^2}{7}+\frac{2^4}{7^2}+\frac{2^6}{7^3}+\cdots+\frac{2^{2n}}{7^n}\right)$$

$$=\frac{1}{14}\lim_{n\to+\infty}\left(\frac{4}{7}+\frac{4^2}{7^2}+\frac{4^3}{7^3}+\cdots+\frac{4^n}{7^n}\right)$$

$$=\frac{1}{14}\lim_{n\to+\infty}\left[\frac{4}{7}+\left(\frac{4}{7}\right)^2+\left(\frac{4}{7}\right)^3+\cdots+\left(\frac{4}{7}\right)^n\right]$$, which is an infinite geometric series.

$$=\frac{1}{14}\frac{\frac{4}{7}}{1-\frac{4}{7}}=\underline{\underline{\frac{2}{21}}}$$

9. Show that the limit of the sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$ exists.

$$\frac{a_{n+1}}{a_n}=\frac{\frac{n+1}{2n+3}}{\frac{n}{2n+1}}=\frac{(n+1)(2n+1)}{(n)(2n+3)}=\frac{(2n^2+3n)+1}{2n^2+3n}>1$$

$\therefore a_{n+1}>a_n$, $\therefore$ The sequence a_n is monotonic increasing.

Note: Strictly increasing implies monotonic increasing, as montonic increasing means ">**or**=". Monotonic increasing does not imply strictly increasing.

$\frac{n}{2n+1}<\frac{2n+1}{2n+1}=1$, the sequence is upper bounded.

By the Monotone bounded theorem, $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$ exists.

10. Find $\lim_{x\to-\infty}\frac{3x+1}{\sqrt{4x^2-1}}$.

Let $L=\lim_{x\to-\infty}\frac{3x+1}{\sqrt{4x^2-1}}$

Put $y = -x$. When $x\to-\infty, y\to+\infty$.

$$L=\lim_{y\to+\infty}\frac{3(-y)+1}{\sqrt{4(-y)^2-1}}=\lim_{y\to+\infty}\frac{-3y+1}{\sqrt{4y^2-1}}=\lim_{y\to+\infty}\frac{-3+\frac{1}{y}}{\sqrt{4-\frac{1}{y^2}}}=\frac{-3+0}{\sqrt{4-0}}=-\mathbf{\frac{3}{2}}$$

11. Use L'hopital Rule to evaluate $\lim_{x\to\infty}\left(\sqrt{x^2+6x}-x\right)$.

Method 1

As $x\to-\infty, \lim_{x\to\infty}\left(\sqrt{x^2+6x}-x\right)=+\infty$. We take $x\to+\infty$, and we assume $x>0$.

$$\lim_{x\to\infty}\left(\sqrt{x^2+6x}-x\right)=\lim_{x\to\infty}\left(x\sqrt{1+\frac{6}{x}}-x\right)=\lim_{x\to\infty}\left(\frac{\sqrt{1+\frac{6}{x}}-1}{\frac{1}{x}}\right),\quad \frac{0}{0}\text{ form}$$

$$=\lim_{x\to\infty}\left(\frac{-\frac{3}{\sqrt{1+\frac{6}{x}}x^2}}{-\frac{1}{x^2}}\right)=\lim_{x\to\infty}\left(\frac{3}{\sqrt{1+\frac{6}{x}}}\right)=3\quad\text{(by L'hopital Rule)}$$

Method 2

Put $y=\frac{1}{x}$, as $x\to+\infty, y\to0^+$.

$$\lim_{x\to+\infty}\left(\sqrt{x^2+6x}-x\right)=\lim_{y\to0^+}\left(\sqrt{\frac{1}{y^2}+\frac{6}{y}}-\frac{1}{y}\right)=\lim_{y\to0^+}\left(\frac{\sqrt{1+6y}-1}{y}\right)\quad,\left(\ \frac{0}{0}\text{ form }\right)$$

$$=\lim_{y\to0^+}\left(\frac{\frac{6}{2\sqrt{1+6y}}}{1}\right)==\lim_{y\to0^+}\left(\frac{3}{\sqrt{1+6y}}\right)=\frac{3}{\sqrt{1+6(0)}}=3\ .$$

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